Heat kernel estimates for boundary trace of reflected diffusions

Mathav Murugan

The University of British Columbia

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Joint work with Naotaka Kajino (Kyoto University).

Brownian motion, Bessel and symmetric Stable processes.

- Let X_t denote the Brownian motion on $\mathbb{R}^n$.
- For $\alpha \in (0,2)$, S_t denote the symmetric α-stable process. That is, (S_t) has independent stationary increments with characteristic function $\mathbb{E}[\exp(i\xi \cdot S_t)] = \exp(-t\,\|\xi\|^{\alpha})$.
- Let Y_t denote the $(2-\alpha)$-dimensional Bessel process on $[0,\infty)$ generated by $\frac{d}{dy^2} + \frac{1-\alpha}{y}\frac{d}{dy}$ independent of X_t.
- Let L_t denote the local time at zero of the process Y_t and let τ_t denote the right continuous inverse of L_t defined as

$$\tau_t = \inf\{s > 0 : L_s > t\}.$$

- **Theorem** (Molchanov, Ostrovskii '69, Caffarelli, Silvestre '07)

$$(X_{\tau_t}, Y_{\tau_t}) \overset{(law)}{=} (S_t, 0).$$

- The case $\alpha = 1$ was shown earlier by Spitzer '58.

Analytic significance due to Caffarelli-Silvestre

- The process $Z_t = (X_t, Y_t)$ is a degenerate diffusion on the upper half space $\mathbb{H}^{n+1} = \mathbb{R}^n \times [0, \infty)$ with generator

$$L_\alpha = \Delta_x + \partial_{yy} + \frac{1-\alpha}{y}\partial_y.$$

- Key observation due to Caffarelli and Silvestre: properties of the non-local fractional Laplace operator $-(-\Delta)^{\alpha/2}$ in $\mathbb{R}^n$ can be deduced from corresponding properties of the local operator L_α on $\mathbb{H}^{n+1}$.

Caffarelli-Silvestre version

- The diffusion $Z_t = (X_t, Y_t)$ on $\mathbb{H}^{n+1}$ corresponds to the Dirichlet form $L^2(\mathbb{H}^{n+1}, y^{1-\alpha}\, dy\, dx)$ given by

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^n} \int_{(0,\infty)} |\nabla u(x,y)|^2\, y^{1-\alpha} dy\, dx.$$

- The Dirichlet form corresponding to the boundary trace process of the diffusion Z_t on $\partial\mathbb{H}^{n+1} \equiv \mathbb{R}^n$ is given by

$$\check{\mathcal{E}}(f,f) = \int_{\mathbb{R}^n} \int_{(0,\infty)} |\nabla u(x,y)|^2\, y^{1-\alpha} dy\, dx$$

  where u solves the Dirichlet problem (harmonic for $\mathcal{E}$)

  $L_\alpha u \equiv 0, \quad$ on $\mathbb{R}^n \times (0,\infty)$, with boundary value $u(x,0) = f(x)$.

Caffarelli-Silvestre version

- Caffarelli-Silvestre show that

$$\int_{\mathbb{R}^n}\int_{(0,\infty)} |\nabla u(x,y)|^2 y^{1-\alpha} dy\, dx = \int_{\mathbb{R}^n} [(-\Delta)^{\alpha/2} f](x) f(x)\, dx$$

$$= c_{n,\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))^2}{|x-y|^{n+\alpha}}\, dy\, dx,$$

where u solves the Dirichlet problem (harmonic function with prescribed boundary value)

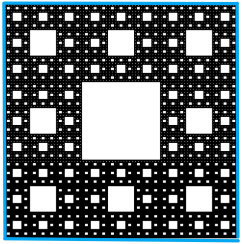
$L_\alpha u \equiv 0, \quad$ on $\mathbb{R}^n \times (0,\infty)$, with boundary value $u(x,0) = f(x)$,

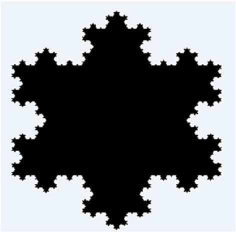
$\mathcal{F}\left[(-\Delta)^{\alpha/2} f\right](\xi) = |\xi|^\alpha \mathcal{F}(f)$ denotes the fractional Laplacian, and $\mathcal{F}[\cdot]$ denotes the Fourier transform.

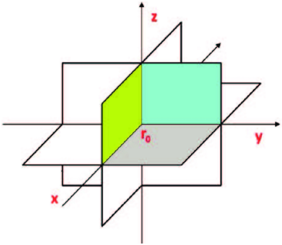
- This is the Dirichlet form of the symmetric α-stable process.

Questions addressed in this work

- Does the boundary trace process behave like a symmetric stable process for other domains and diffusions?
- For example, what if we replace the reflected Brownian motion with a diffusion generated by uniformly elliptic operator?
- What if we consider reflected Brownian motion on non-smooth domains like Lipschitz domains (first orthant) or uniform domain (snowflake domain)?
- What if we consider the trace of the Brownian motion on the Sierpiński carpet on its outer square boundary?







Trace of reflected Brownian motion on smooth domains

- (Osborn '60, Douglas '31) If U is a smooth domain in $\mathbb{R}^n$ with Green function $g_U(\cdot,\cdot)$ and u is harmonic in U with prescribed boundary value $f : \partial U \to \mathbb{R}$, then the Dirichlet energy of u can be expressed in terms of f as

$$\int_U |\nabla u|^2 (x)\, dx = \frac{1}{2} \int_{\partial U} \int_{\partial U} (f(\xi) - f(\eta))^2 \frac{\partial^2 g_U(\xi,\eta)}{\partial \vec{n}_\xi \partial \vec{n}_\eta}\, d\sigma(\xi)\, d\sigma(\eta),$$

  where σ is the surface measure on ∂U and $\vec{n}_\xi, \vec{n}_\eta$ are inward pointing unit normal vectors at $\xi, \eta \in \partial U$.
- The proof involves integration by parts (Gauss-Green formula).
- How to handle non-smooth domains like the snowflake domain?
- Doob '62 proved a remarkable formula for the Dirichlet energy for domains that are not necessarily smooth. We call it the Doob-Naïm formula.

Naïm kernel

- Let U denote a transient domain and let $\partial_M U$ denote its Martin boundary. Let $x_0 \in U$ be a base point.
- The Naïm kernel is defined as

$$\Theta_{x_0}^U(x,y) = \frac{g_U(x,y)}{g_U(x_0,x)g_U(x_0,y)}, \quad \text{for } x,y \in U \setminus \{x_0\}.$$

- Naïm '57 showed that the above function can be extended continuously to the Martin boundary $\partial_M U$ with respect to Cartan's fine topology on $U \cup \partial_M U$ as

$$\Theta_{x_0}^U(\xi,\eta) = \lim_{x\to\xi}\lim_{y\to\eta} \frac{g_U(x,y)}{g_U(x_0,x)g_U(x_0,y)}, \quad \text{for } \xi,\eta \in \partial_M U, \xi \neq \eta,$$

The above limits are with respect to Cartan's fine topology.

Doob-Naïm formula

Theorem (Doob '62)

Let $\omega_{x_0}^U(\cdot)$ denote the harmonic measure for the Brownian motion on the Martin boundary $\partial_M U$, where $x_0 \in U$ is the starting point. The Dirichlet energy of a harmonic function on $u : U \to \mathbb{R}$ with a prescribed boundary value $f : \partial_M U \to \mathbb{R}$ is given by

$$\int_U |\nabla u|^2 \, dx = \int_{\partial_M U} \int_{\partial_M U} (f(\xi) - f(\eta))^2 \, \Theta_{x_0}^U(\xi, \eta) \, d\omega_{x_0}^U(\xi) \, d\omega_{x_0}^U(\eta).$$

In other words, the jump kernel of the boundary trace of reflected Brownian motion with respect to the harmonic measure is the Naïm kernel.

Doob's proof relies on a version of integration by parts that uses the notion of fine normal derivatives introduced by Naïm '57.

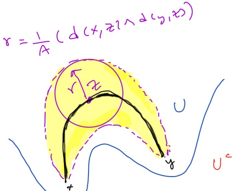
Uniform domains and capacity density condition

- A domain U is said to satisfy the capacity density condition if there exists C_1, A_1 such that for all $\xi \in \partial U, 0 < r \,\mathrm{diam}(U)/A_1$,

$$\mathrm{Cap}(B(\xi, r), B(\xi, 2r)^c) \le C_1 \,\mathrm{Cap}(B(\xi, r) \setminus U, B(\xi, 2r)^c)$$

- (Martio, Sarvas '79) A connected, non-empty, proper open set $U \subsetneq X$ is said to be a uniform domain if there exists $A > 1$ such that for every pair of points $x, y \in U$, there exists a curve γ in U from x to y such that its diameter $\mathrm{diam}(\gamma) \le A d(x, y)$, and

$$\mathrm{dist}\,(z, U^c) \ge A^{-1} \min\,(d(x, z), d(y, z)) \quad \text{for all } z \in \gamma.$$



Why study uniform domains?

- There is a one-to-one correspondence between a class of uniform domains and Gromov-hyperbolic spaces (Bonk, Heinonen, Koskela '01)
- (Rajala '21) Uniform domains are abundant in the sense that every bounded domain can be approximated by a uniform domain.
- Given a complete, doubling metric space (X, d) that is bi-Lipschitz equivalent to a length space, a bounded domain Ω and $\epsilon > 0$, there exist uniform domains Ω_o and Ω_i such that

$$\Omega_i \subset \Omega \subset \Omega_0, \quad \Omega_o \subset [\Omega]_\epsilon, \quad (\Omega_i)^c \subset [\Omega^c]_\epsilon.$$

- Uniform domains can have fractal boundaries and are far from smooth in general.

Our version of Doob-Naïm formula

Theorem 1 (Kajino, M. '24+): Let (X, d, m) be a metric measure space and let $(\mathcal{E}, \mathcal{F})$ denote a Dirichlet form on $L^2(m)$ corresponding to a diffusion process on X that satisfies sub-Gaussian heat kernel estimates. Let U be a uniform domain satisfying the capacity density condition and $x_0 \in U$. Then

(a) The Naïm kernel has a continuous extension to the topological boundary as a (jointly) continuous function on $\bigl((\overline{U} \setminus \{x_0\}) \times (\overline{U} \setminus \{x_0\})\bigr) \setminus (\overline{U} \setminus \{x_0\})_{\mathrm{diag}}$.

(b) The Dirichlet energy of a harmonic function in U (with respect to $(\mathcal{E}, \mathcal{F})$) with prescribed boundary value $f : \partial U \to \mathbb{R}$ is

$$\int_{\partial U} \int_{\partial U} (f(\xi) - f(\eta))^2 \, \Theta_{x_0}^{U}(\xi, \eta) \, d\omega_{x_0}^{U}(\xi) \, d\omega_{x_0}^{U}(\eta),$$

where $\omega_{x_0}^{U}$ is the harmonic measure on ∂U for the diffusion starting at x_0 and stopped upon exiting U.

Remarks on the Doob-Naïm formula

- Our proof relies on boundary Harnack principle which was not available to Doob in 1962 as the first versions of BHP were only developed in the late seventies.
- We use the boundary Harnack principle established by A. Chen '24+. following the work of Aikawa '01.
- The degenerate diffusion of Molchanov and Ostrovskii satisfies the condition of our theorem due to earlier results of Fabes, Kenig, Serapioni '82 and results of Grigor'yan '91, Saloff-Coste '92. Therefore the result of Caffarelli-Silvestre follows from our version of the Doob-Naïm formula.

Further remarks on the Doob-Naïm formula

- Doob's approach only is available for the Brownian motion but can handle more general domains.
- Naïm's formula for fine normal derivative is not available for uniformly elliptic operators in $\mathbb{R}^n$.
- The Doob-Naïm formula is new for elliptic operators on $\mathbb{R}^n$ given by $Lf := \mathrm{div}\,(A(\cdot)\nabla f)$, where A is a measurable $n \times n$-positive definite matrix valued whose eigenvalues are uniformly bounded from above and below.
- There is a diffusion generated by uniformly elliptic operator on two dimensional upper half-space with singular harmonic measure (with respect to the surface measure) due to Caffarelli, Fabes, Kenig '81 based on a quasi-conformal mapping of Ahlfors-Beurling '56.

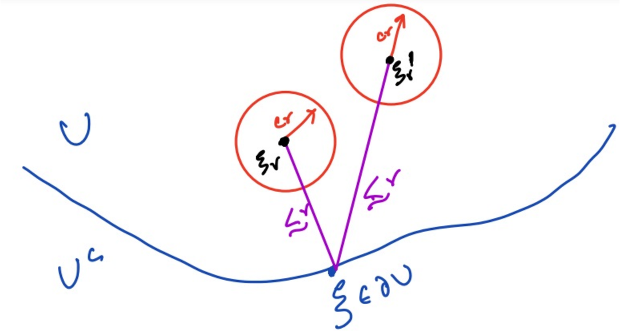
Estimates on harmonic measure

Theorem 2 (Kajino, M. '24+) Under the assumptions of Theorem 1, there exist $C, A \in (0, \infty)$ such that for any $x_0 \in U, \xi \in \partial U, r < d(x_0, \xi)/A$

$$C^{-1} \frac{g_U(x_0, \xi_r)}{g_U(\xi_r', \xi_r)} \leq \omega_{x_0}^U(B(\xi, r)) \leq C \frac{g_U(x_0, \xi_r)}{g_U(\xi_r', \xi_r)}$$

where $\xi_r, \xi_r' \in U$ are chosen so that

$$\operatorname{dist}(\xi_r, U^c) \asymp \operatorname{dist}(\xi_r', U^c) \asymp d(\xi_r, \xi) \asymp d(\xi_r', \xi) \asymp d(\xi_r, \xi_r') \asymp r.$$



Boundary local time

For the Molchanov-Ostrowski diffusion $Z_t = (X_t, Y_t)$, the local time L_t of Y_t at zero serves as a 'boundary local time' with

$$\mathbb{R}^n \times \{0\} = \partial\mathbb{H}^{n+1} = \{z \in \mathbb{H}^{n+1} : \mathbb{P}_z[L_t > 0] = 1 \text{ for all } t > 0\}.$$

L_t is a positive continuous additive functional associated with the Markov process Z_t supported on $\partial\mathbb{H}^{n+1}$.

A positive continuous additive functional (PCAF) associated $A_t : \Omega \to [0, \infty), t \geq 0$ with a Markov process $(\Omega, \mathcal{F}_t, Z_t, \theta_t, \mathbb{P}_z)$ where $\mathcal{F}_t$ is the associated filtration, $\theta_t : \Omega \to \Omega$ is the time-shift operator such that

(a) (positive and continuous) $t \mapsto A_t(\omega)$ is non-negative, continuous and $A_0(\omega) = 0$ $\mathbb{P}_z$-almost surely for all z.

(b) (adapted) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable.

(c) (additive property) $A_{t+s}(\omega) = A_t(\omega) + A_s(\omega \circ \theta_t)$ $\mathbb{P}_z$-almost surely for all z.

Boundary local time of a reflected diffusion

Let X_t be a diffusion on $\overline{U}$ symmetric with respect to the measure m and let (A_t) be a positive continuous additive function.
Theorem (Revuz '70): There exists a unique measure μ on $\overline{U}$ such that for all non-negative measurable function $h, f : \overline{U} \to [0, \infty)$, we have

$$\mathbb{E}_{h \cdot m} \left[\int_0^t f(X_s)\, dA_s \right] = \int_0^t \int_{\overline{U}} f(y) P_s h(y)\, \mu(dy)\, ds,$$

where $P_s h(y) = \mathbb{E}_y[h(X_s)]$ is the corresponding Markov semigroup.
Conversely, given any smooth measure μ on $\overline{U}$ there exists a unique PCAF that satisfies the above property.
Examples: Let A_t be the local time at $x \in \mathbb{R}$ for Brownian motion on $\mathbb{R}$, then $\mu = \delta_x$.
If $A_t = \int_0^t f(X_s)\, ds$, where $f \geq 0$, then $\mu = f \cdot m$.

Boundary measure on bounded domains

By the Revuz correspondence, the boundary local time can be defined by choosing a suitable measure μ supported on the boundary ∂U.
The harmonic measure $\omega_{x_0}^U$ is a natural choice for μ.
For bounded domains, we choose $\mu = \omega_{x_0}^U$, where $x_0 \in U$ is chosen so that $\operatorname{dist}(x_0, U^c) \asymp \operatorname{diam}(U)$.
This choice defines the measure up to a bounded perturbation, since if $x_0, x_0' \in U$ satisfy the above conditions

$$\frac{d\omega_{x_0'}^U}{d\omega_{x_0}^U} \asymp 1 \quad \text{on } \partial U.$$

Boundary measure on unbounded domains

For unbounded domains, there is a 'canonical' variant of harmonic measure due to Kenig and Toro '99 for non-tangentially accessible domains in $\mathbb{R}^n$.

For each $x_0 \in U$, there exists a unique measure $\mu_{x_0}^U$ on ∂U such that

$$\mu_{x_0}^U(\cdot) = \lim_{y \to \infty} \frac{1}{g(x_0, y)} \omega_y^U(\cdot)$$

For $x_0, x_1 \in U$, there exists $c_{x_0,x_1} \in (0, \infty)$ such that $\mu_{x_1}^U(\cdot) = c_{x_0,x_1} \mu_{x_0}^U(\cdot)$.

We choose the boundary measure $\mu = \mu_{x_0}^U$ for some $x_0 \in U$ in the case of unbounded domains.

Theorem 2 implies that μ is doubling (for both bounded and unbounded domains).

Example: For the Molchanov-Ostrovskii diffusion μ is (a positive multiple of) the Lebesgue measure on $\partial \mathbb{H}^{n+1} = \mathbb{R}^n$.

The boundary trace process

Theorem 3 (Kajino, M. '24+): Let U be a domain as in Theorem 1 and let $(X_t)_{t\geq 0}$ be reflected diffusion on $\overline{U}$.
Let A_t denote the positive continuous additive functional corresponding to the boundary measure μ. Then A_t has support ∂U; that is

$$\partial U = \{x \in \overline{U} : \mathbb{P}_x[A_t > 0] = 1 \text{ for all } t > 0\}.$$

Then $\check{X}_t = X_{\tau_t}$, where τ_t is the right continuous inverse of A_t is a μ-symmetric Markov process on ∂U whose Dirichlet form on $L^2(\partial U, \mu)$ is given by the Doob-Naïm formula.
The jump process $\check{X}_t$ on ∂U satisfies stable-like heat kernel bounds.

Properties of α-stable process

Let us express properties of the α-stable process in terms of the Lebesgue measure m, Euclidean distance $d(\cdot, \cdot)$ and the space-time scaling function $\phi(r) = r^\alpha$.
The jump kernel and exit time bounds are

$$J(x,y) \asymp \frac{1}{m(B(x,d(x,y)))\phi(d(x,y))}, \quad \mathbb{E}_x[\tau_{B(x,r)}] \asymp \phi(r),$$

and the heat kernel bound is

$$p_t(x,y) \asymp \frac{1}{m(B(x,\phi^{-1}(t)))} \wedge \frac{t}{m(B(x,d(x,y)))\phi(d(x,y))}.$$

for all $x, y \in \mathbb{R}^n$, $t, r > 0$.

Space-time scaling of the boundary trace process

$$\Phi(\xi, r) \asymp \begin{cases} g_U(\xi_r, x_0), & \text{for all } \xi \in \partial U, 0 < r < \operatorname{diam}(U)/A < \infty, \\ h^U_{x_0}(\xi_r), & \text{for all } \xi \in \partial U, r > 0, \text{ if } \operatorname{diam}(U) = \infty, \end{cases}$$

where $\xi_r \in U$ satisfies $d(\xi_r, \xi) \asymp \operatorname{dist}(\xi_r, U^c) \asymp r$ and $h^U_{x_0}(\cdot)$ is the unique positive harmonic function on U with zero boundary condition with normalization $h^U_{x_0}(x_0) = 1$.

Examples:

For the Molchanov-Ostrovskii diffusion, the positive harmonic function is $h(x, y) = y^\alpha$ for $x \in \mathbb{R}^n, y \in [0, \infty)$.

For the reflected Brownian motion on the first orthant $\overline{U} = [0, \infty)^n$, the harmonic function is

$$h(x) = \prod_{i=1}^{n} x_i, \quad h(\xi_r) = \prod_{i=1}^{n} (\xi_i + r),$$

where $x = (x_1, \ldots, x_n) \in \overline{U}, \xi = (\xi_1, \ldots, \xi_n) \in \partial U$.

Estimates for the boundary trace process

The jump kernel and exit time bounds are

$$J(\xi,\eta) \asymp \frac{1}{\mu(B(\xi,d(\xi,\eta)))\Phi(\xi,d(\xi,\eta))}, \quad \mathbb{E}_\xi[\tau_{B(\xi,r)}] \asymp \Phi(\xi,r),$$

and the heat kernel bound is

$$p_t(\xi,\eta) \asymp \frac{1}{\mu(B(\xi,\Phi^{-1}(\xi,t)))} \wedge \frac{t}{\mu(B(\xi,d(\xi,\eta)))\Phi(\xi,d(\xi,\eta))}.$$

for all $x, y \in \overline{U}, 0 < r < C^{-1}\operatorname{diam}(U)$, where $\Phi^{-1}(\xi,t)$ is the inverse of $r \mapsto \Phi(\xi,r)$ evaluated at t.

Ingredients in the proof of Theorem 3

- The jump kernel estimates follow from the Doob-Naïm formula (Theorem 1) and estimates on harmonic measure (Theorem 2).
- The reflected diffusion on $\overline{U}$ inherits sub-Gaussian heat kernel estimates from the ambient space (M. '24).
- This implies green function bounds on the boundary trace process which in turn implies exit time bounds for the boundary trace process.
- General conditions on stable-like heat kernel bounds due to Chen-Kumagai-Wang '21 and Grigor'yan-Hu-Hu '23 allows us to deduce stable-like heat kernel bounds from jump kernel and exit time estimates.
- Our approach uses heat kernel estimates for the reflected diffusion to obtain heat kernel estimates for the corresponding boundary trace process (similar to Caffarelli-Silvestre).

Thank you for your attention

N. Kajino, M. Murugan. Heat kernel estimates for boundary trace of reflected diffusions on uniform domains. arXiv:2312.08546.

M. Murugan, Heat kernel for reflected diffusion and extension property on uniform domains, *Probab. Theory Related Fields* (2024).